curve $y = f(x)$ at the point x_0.

For $k = {}^3/_8$ the characteristic exponents λ_1 and λ_2 of the linearized system of perturbed-motion equations are related to the third order resonance relationship $\lambda_1 = 3\lambda_2$. Calculations show that if $f'''(x_0) \neq 0$ here, then the periodic motion under investigation is unstable.

For the remaining values of k in the interval $(0, {}^1/_2)$ the solution of the stability question depends on the parameters

$$\varkappa_1 = \frac{f'''(x_0)}{[f''(x_0)]^2}, \quad \varkappa_2 = \frac{f^{IV}(x_0)}{[f''(x_0)]^3}$$

For $k \neq {}^1/_4$ (there is no fourth-order resonance $\lambda_1 = 4\lambda_2$), orbital stability of the periodic solutions under consideration hold in the general case of non-degeneracy of the normal form.

In particular, calculations performed for the parabola $(\varkappa_1 = \varkappa_2 = 0)$ and the sinusoid $(\varkappa_1 = 0,\ \varkappa_2 = -1)$ show that the solutions (2.4) are stable for these curves for all values of k in the interval $(0, {}^1/_2)$.

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SOME CONDITIONS FOR THE EXISTENCE AND STABILITY OF PERIODIC OSCILLATIONS IN NON-LINEAR NON-AUTONOMOUS HAMILTONIAN SYSTEMS*

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The sufficient conditions for the existence and uniqueness of periodic solutions are obtained for non-autonomous Hamiltonian systems by the method of continuation with respect to the parameter /1/ (similar results were established for certain vector equations by other methods in /2, 3/). Using the theorem on the directed width of stability regions /4/, stability criteria to a first approximation of these solutions are obtained. The effect of small dissipative forces on stability is investigated. Systems are considered in which some of the generalized coordinates are angular. The conditions for the existence, uniqueness, and stability are obtained, as well as the upper bounds of solutions that correspond to periodic rotational motions of the angular coordinates with any preassigned average velocities that are multiples of the perturbing effect. The periodic oscillatory and rotational motions of two coupled pendulums are considered, as an example.

1. We consider the system

$$x_i^{\cdot} = \frac{\partial H}{\partial x_{i+n}}, \quad \dot{x}_{i+n} = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n \tag{1.1}$$

where $x_1, \ldots, x_n$ are the generalized coordinates, $x_{n+1}, \ldots, x_{2n}$ are the momenta, and the Hamiltonian function $H(x_1, \ldots, x_{2n}, \omega t)$ is doubly differentiable with respect to x_i and 2π-periodic in ωt.

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The variational equation

$$Jy^{\cdot} = A(\mathbf{x}(t), \omega t)\,\mathbf{y} \tag{1.2}$$

$$J = \begin{Vmatrix} 0 & -I_n \\ I_n & 0 \end{Vmatrix}, \quad A(\mathbf{x}, \omega t) = \| a_{ik}(\mathbf{x}, \omega t) \|, \quad a_{ik} = \frac{\partial^2 H}{\partial x_i \partial x_k}$$

where I_n is a unit matrix of order n, corresponds to the solution $\mathbf{x}(t)$ of system (1.1).

Suppose that the matrix $A(\mathbf{x}, \omega t)$ is of fixed sign for all $\mathbf{x}, t$ and its elements are bounded in R^{2n+1}. Then we can find constant matrices A_- and A_+ of fixed sign, of the same sign as $A(\mathbf{x}, \omega t)$, and such that $A_- \leqslant A(\mathbf{x}, \omega t) \leqslant A_+$, i.e. for any vector of order and any the respective quadratic forms satisfy the inequality

$$(\mathbf{c}, A_-\mathbf{c}) \leqslant (\mathbf{c}, A(\mathbf{x}, \omega t)\,\mathbf{c}) \leqslant (\mathbf{c}, A_+\mathbf{c}) \tag{1.3}$$

The eigenvalues of the matrices $J^{-1}A_-$ and $J^{-1}A_+$ are imaginary by virtue of the fact that A_- and A_+ are of fixed sign; we denote them by $\pm i\omega_k^-$ and $\pm i\omega_k^+$, respectively, where $(k = 1, \ldots, n)$.

Theorem 1. If

$$\omega \notin \frac{1}{m}[\omega_k^-, \omega_k^+], \quad k = 1, \ldots, n; \quad m = 1, 2, \ldots \tag{1.4}$$

system (1.1) has a unique periodic solution with period $T = 2\pi/\omega$.

This solution is stable to a first approximation, when the condition

$$\omega \notin \frac{1}{m}[\omega_i^- + \omega_k^-, \omega_i^+ + \omega_k^+], \quad i, k = 1, \ldots, n; \quad m = 1, 2, \ldots \tag{1.5}$$

is satisfied.

Proof. We set $H = H(\mathbf{x}, \omega t, \varepsilon) = (\mathbf{x}, A_-\mathbf{x}) + \varepsilon[H(x, \omega t) - (\mathbf{x}, A_-\mathbf{x})]$ in (1.1). We will show that when $\varepsilon \in [0, 1]$ the following conditions are satisfied. All solutions $x(t, \varepsilon)$ are continuable in $(0, \infty)$ and any periodic solution of period T has the estimate $(\mathbf{x}(0, \varepsilon), \mathbf{x}(0, \varepsilon)) \leqslant N$ which is independent of ε; the respective equation (1.2) has no periodic solutions of period T, and when $\varepsilon = 0$ it has a periodic solution of period T. As shown in /1/, the given system (corresponding to $\varepsilon = 1$) under these conditions has a periodic solution of period T.

Let us represent (1.1) in the form

$$J\mathbf{x}^{\cdot} = H_{\mathbf{x}}(\mathbf{x}, \omega t, \varepsilon) = \int_0^1 H_{\mathbf{x}\theta}(\theta\mathbf{x}, \omega t, \varepsilon)\,d\theta + H_{\mathbf{x}}(\theta\mathbf{x}, \omega t, \varepsilon)|_{\theta=0} = \int_0^1 H_{\mathbf{x}\mathbf{x}}(\theta\mathbf{x}, \omega t, \varepsilon)\,\mathbf{x}\,d\theta + \varepsilon H_{\mathbf{x}}(0, \omega t)$$

Thus any solution $\mathbf{x}(t, \varepsilon)$ of system (1.1) also satisfies the equation

$$J\mathbf{x}^{\cdot} = A_*(t, \varepsilon)\,\mathbf{x} + \varepsilon\mathbf{f}(t) \tag{1.6}$$

$$A_*(t, \varepsilon) = \int_0^1 A(\theta\mathbf{x}(t, \varepsilon), \omega t, \varepsilon)\,d\theta, \quad \mathbf{f}(t) = \mathbf{f}(t + T) = H_{\mathbf{x}}(0, \omega t)$$

and since $A_- \leqslant A(\mathbf{x}, \omega t, \varepsilon) \leqslant A(\mathbf{x}, \omega t) \leqslant A_+$, we have $A_- \leqslant A_*(t, \varepsilon) \leqslant A_+$ when $\varepsilon \in [0, 1]$.

We set $\varphi(t, \varepsilon) = (\mathbf{x}(t, \varepsilon), \mathbf{x}(t, \varepsilon))$. By virtue of (1.6)

$$\varphi^{\cdot} = 2(\mathbf{x}, \mathbf{x}^{\cdot}) = 2(\mathbf{x}J^{-1}A_*\mathbf{x} + \varepsilon J^{-1}\mathbf{f})$$

hence, using the Cauchy inequality, we obtain

$$\varphi^{\cdot} \leqslant 2(\lambda_+\varphi + c\varphi^{1/2}), \quad c = \max(\mathbf{f}(t), \mathbf{f}(t))^{1/2}$$

where λ_+ is the largest eigenvalue of matrix A_+. Using Chaplygin's theorem on differential inequalities, we obtain

$$\varphi(t, \varepsilon) \leqslant \frac{1}{\lambda_+^2}[(\lambda_+\varphi^{1/2}(0, \varepsilon) + c)\exp(\lambda_+ t) - c]^2 \ \text{for} \ t > 0 \tag{1.7}$$

It follows from (1.7) that the solution $\mathbf{x}(t, \varepsilon)$ is continuable in $(0, \infty)$. Consider the selfconjugate boundary value problem

$$J\mathbf{y}^{\cdot} = \lambda Q(t)\,\mathbf{y}, \ \mathbf{y}(0) = \mathbf{y}(T) \tag{1.8}$$

Let $\lambda_i, \lambda_i^-, \lambda_i^+$ $(i = 1, 2, \ldots)$ be the positive eigenvalues of the boundary value problem (1.8) arranged in ascending order, when $Q = A(\mathbf{x}(t, \varepsilon), \omega t)$, $Q = A_-$ and $Q = A_+$, respectively. Since the inequality $A_- \leqslant A(\mathbf{x}, \omega t) \leqslant A_+$ holds when $\varepsilon \in [0, 1]$, then $\lambda_i^+ \leqslant \lambda_i \leqslant \lambda_i^-$ /5/. The spectra λ^- and λ^+ are obviously formed from quantities $m\omega/\omega_k^-$ and $m\omega/\omega_k^+$ $(k = 1, \ldots, n;\ m = 0, \pm 1, \pm 2, \ldots)$, we have under condition (1.4) $\lambda_i \neq 1$. Consequently, when $\varepsilon \in [0, 1]$ equation (1.2), which corresponds to any periodic solution $\mathbf{x}(t, \varepsilon)$, does not have T-periodic solutions.

When $Q = A_*(t, \varepsilon)$, the eigenfunctions $\mathbf{y}_k(t, \varepsilon)$ of problem (1.8) form a complete system (by virtue of $A_*(t, \varepsilon) > 0$) and $A_* y_i$ and y_k are orthogonal in the interval $[0, T]$ when $i \neq k$ /5/. Assuming that $y_k(t, \varepsilon)$ are appropriately normalized, we represent the periodic solution of (1.6) in the form

$$\mathbf{x}(t, \varepsilon) = \varepsilon \sum_{k=-\infty}^{\infty} \frac{f_k(\varepsilon)\, \mathbf{y}_k(t, \varepsilon)}{\lambda_k(\varepsilon) - 1}, \qquad f_k(\varepsilon) = \int_0^T (\mathbf{f}(t), \mathbf{y}_k(t, \varepsilon))\, dt \tag{1.9}$$

where $\lambda_k(\varepsilon)$ are the eigenvalues of problem (1.8), when $Q = A_*(t, \varepsilon)$; $\lambda_k(\varepsilon) > 0$ when $k > 0$ and $\lambda_k(\varepsilon) \leqslant 0$ when $k \leqslant 0$.

Let λ_- be the minimum eigenvalue of the matrix A_- and $\delta = \min(1, |\lambda_k^- - 1|, |\lambda_k^+ - 1|)$ when $k > 0$. Taking into account that $A_*(t, \varepsilon) \geqslant A_-$, $\lambda_k(\varepsilon) \in [\lambda_k^+, \lambda_k^-]$ and using (1.9), we find that when $\varepsilon \in [0, 1]$

$$\int_0^T \varphi\, dt \leqslant \frac{1}{\lambda_-} \int_0^T (A_* \mathbf{x}, \mathbf{x})\, dt = \frac{\varepsilon^2}{\lambda_-} \sum_{k=-\infty}^{\infty} \frac{f_k^2(\varepsilon)}{(\lambda_k(\varepsilon) - 1)^2} <$$
$$\frac{1}{\lambda_- \delta^2} \sum_{k=-\infty}^{\infty} f_k^2(\varepsilon) = \frac{1}{\lambda_- \delta^2} \int_0^T (A_*^{-1}\mathbf{f}, \mathbf{f})\, dt \leqslant \frac{1}{\lambda_-^2 \delta^2} \int_0^T (\mathbf{f}, \mathbf{f})\, dt = N_1 \tag{1.10}$$

where the constant N_1 is independent of ε. By virtue of (1.7) and (1.10) $\varphi(0, \varepsilon)$ also has an upper bound that is independent of ε.

When $\varepsilon = 0$ system (1.1) has the periodic solution $x(t, 0) = 0$ of period T. Thus the conditions mentioned above for periodic solutions $x(t)$ of period T to exist are satisfied.

Under the conditions indicated here any periodic solution $\mathbf{x}(t, \varepsilon)$ can be continued in a unique way with respect to ε in $[0, 1]$ /1/, hence the number of periodic solutions in the same for any $\varepsilon \in [0, 1]$. Since $\omega \neq \omega_k^-/m$ $(k = 1, \ldots, n;\ m = 1, 2, \ldots)$, the solution $\mathbf{x}(t, 0)$ and consequently $\mathbf{x}(t, 1) = \mathbf{x}(t)$ are unique.

We denote the eigenvalues of the matrix $J^{-1}(A_- + s(A_+ - A_-))$ by $\pm i\omega_k(s)$. Since $\omega_k(s) \in [\omega_k^-, \omega_k^+]$ when $s \in [0, 1]$, under condition (1.5) we have $(\omega_k(s) + \omega_i(s))\, T \neq 2\pi m$ $(i, k = 1, \ldots, n;\ m = 1, 2, \ldots)$. Therefore the equation

$$J\mathbf{x}^{\cdot} = (A_- + s(A_+ - A_-))\, \mathbf{x}$$

for any $s \in [0, 1]$ is highly stable, i.e. all multiplicators lie on a unit circle and are definite. In agreement with the theorem on the directed width of stability regions /4/, equation (1.2) is also highly stable, i.e. the solution $\mathbf{x}(t)$ is stable to a first approximation. The theorem is proved.

2. Consider the systems defined by the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial K}{\partial x_i^{\cdot}}\right) - \frac{\partial (K - V)}{\partial x_i} = 0, \qquad i = 1, \ldots, n \tag{2.1}$$
$$K(\mathbf{x}, \mathbf{x}^{\cdot}, \omega t) = K(\mathbf{x}, \mathbf{x}^{\cdot}, \omega t + 2\pi), \quad V(\mathbf{x}, \omega t) = V(\mathbf{x}, \omega t + 2\pi)$$

where $K(\mathbf{x}, \mathbf{x}^{\cdot}, \omega t)$ and $V(\mathbf{x}, \omega t)$ are the kinetic and potential energy.

The variational equation

$$[M(t)\, \mathbf{z}^{\cdot} + Q(t)\, \mathbf{z}]^{\cdot} - Q'(t)\, \mathbf{z}^{\cdot} + [U(t) - P(t)]\, \mathbf{z} = 0 \tag{2.2}$$

corresponds to the solution $\mathbf{x}(t)$. Here the prime denotes transposition, and the elements of the matrices M, Q, P and U are equal to the derivatives calculated for $\mathbf{x} = \mathbf{x}(t)$.

$$m_{ik} = \frac{\partial^2 K}{\partial x_i^{\cdot} \partial x_k^{\cdot}}, \qquad q_{ik} = \frac{\partial^2 K}{\partial x_i^{\cdot} \partial x_k}$$
$$p_{ik} = \frac{\partial^2 K}{\partial x_i \partial x_k}, \qquad u_{ik} = \frac{\partial^2 V}{\partial x_i \partial x_k}$$

Equation (2.2) reduces to the form (1.2), and from /4/

$$A = \begin{Vmatrix} U - P + Q'M^{-1}Q & -Q'M^{-1} \\ -M^{-1}Q & M^{-1} \end{Vmatrix}, \qquad \mathbf{y} = \mathbf{z} + M\mathbf{z}^{\cdot} + Q\mathbf{z} \tag{2.3}$$

i.e. the vector $\mathbf{y}$ is equal to the sum of vectors $\mathbf{z}$ and $M\mathbf{z}^{\cdot} + Q\mathbf{z}$.

If the kinetic energy is independent of the coordinates $(K = K(\mathbf{x}^{\cdot}, \omega t))$, then $P = Q = 0$, and inequality (1.3) will be satisfied, if A_- and A_+ are taken as quasidiagonal matrices with elements U_-, M_+^{-1} and U_+, M_-^{-1}, where $0 < U_- \leqslant U \leqslant U_+$, $0 < M_- \leqslant M \leqslant M_+$. As a result, under conditions (1.4) and (1.5), ω_i^- and ω_i^+ are equal to the square roots of the eigenvalues of the matrices $M_+^{-1}U_-$ and $M_-^{-1}U_+$ respectively. Note that the conditions of existence and uniqueness have been obtained for this case by other methods /2, 3/.

Usually $K = {}^1/_2\, (\mathbf{x}^{\cdot}, M(\mathbf{x}, \omega t)\, \mathbf{x}^{\cdot})$ and the matrix $M(\mathbf{x}, \omega t)$ is positive definite for all $\mathbf{x}$ and

t. Moreover, the matrix U in many cases (e.g., in systems with angular coordinates with intensive parametric excitation, etc.) is not positive definite, and the condition $A > 0$ is not satisfied. In the following theorem a weaker condition of positive definiteness in the mean is imposed on the matrix U, i.e.

$$\langle U\rangle = \frac{1}{T}\int_0^T U(\mathbf{x}(t), \omega t)\,dt > 0 \tag{2.4}$$

Let $K = {}^1/_2(\mathbf{x}^\cdot, M(\omega t)\mathbf{x}^\cdot)$, $0 < M_- \leqslant M(\omega t)$ and $U(\mathbf{x}, \omega t) \leqslant U_+$, ω_n^{+2} be the maximum eigenvalue of the matrix $M_-^{-1}U_+$.

Theorem 2. The periodic solution of period T of (2.1), when $\omega > 2\omega_n^+$ under condition (2.4) is stable to a first approximation.

Proof. Since K is independent of $\mathbf{x}$, hence $Q = P = 0$. Consider the equation

$$J\mathbf{y}^\cdot = \lambda A(t)\,\mathbf{y}, \quad A(t) = \begin{Vmatrix} U(\mathbf{x}(t), \omega t) & 0 \\ 0 & M^{-1}(\omega t)\end{Vmatrix} \tag{2.5}$$

Suppose $\rho_i(\lambda)$ and $\mathbf{y}_i(t, \lambda)$ are multipliers and their respective solutions of (2.5), $(\mathbf{y}_i(t + T, \lambda) = \rho_i\mathbf{y}_i(t, \lambda))$. Taking into account that $\mathbf{y}_i = \mathbf{z}_i + M\mathbf{z}_i^\cdot$, we obtain

$$q_i = \int_0^T (A\mathbf{y}_i, \mathbf{y}_i)\,dt = \frac{1}{\lambda}\int_0^T (J\mathbf{y}_i^\cdot, \mathbf{y}_i)\,dt = \frac{2}{\lambda}\int_0^T (M\mathbf{z}_i^\cdot, \mathbf{z}_i^\cdot)\,dt - \frac{1}{\lambda}(M\mathbf{z}_i^\cdot, \mathbf{z}_i)\Big|_0^T \tag{2.6}$$

In view of (2.4) we have $\langle A\rangle > 0$, hence for small λ all multipliers of the first kind are displaced from the point $\rho(0) = 1$ on the upper semicircle, and of the second kind on the lower semicircle /5/. Since the term $|\rho_i| = 1$ outside the integral (2.6) is zero, $q_i > 0$ and, consequently, as $\lambda\ \rho_i(\lambda)$ increases the multipliers continue to move in the same direction /4/. Let λ_* be the the value of λ for which the leading multipliers of various kinds meet at the point $\rho = -1$. Obviously λ_* is the first eigenvalue of the boundary value problem for (2.5) with conditions $\mathbf{y}(0) = -\mathbf{y}(T)$. Since $A(t) < A_+ = \operatorname{diag}(U_+, M_-^{-1})$, then $\lambda_* > \lambda_1^+ = \pi^2 (T\omega_n^+)^{-2} > 1$ when $\omega > 2\omega_n^+$, where λ_1^+ is the first positive eigenvalue of the boundary value problem indicated when $A = A_+$. Consequently, when $\lambda = 1$, all multipliers of the first kind lie on the upper semicircle and of the second kind on the lower semicircle. This proves the stability of solution $\mathbf{x}(t)$.

Remark. If M is a constant matrix $V(\mathbf{x}, \omega t) = V(-\mathbf{x}, \omega t + \pi)$, solutions of the form $\mathbf{x}(t) = -\mathbf{x}(t + T/2)$ are usually considered (this relation is clearly valid, when the periodic solution of period T is unique, since besides $\mathbf{x}(t)$ (2.1) is satisfied by the function $-\mathbf{x}(t + T/2)$). Then $u_{ik}(\mathbf{x}, \omega t) = u_{ik}(-\mathbf{x}, \omega t + \pi)$, $U(t) = U(t + T/2)$, $A(t) = A(t + T/2)$, therefore 2ω must be substituted for ω in the conditions of stability obtained above. In particular Theorem 2 holds when $\omega > \omega_n^+$.

3. Suppose the function $V(\mathbf{x}, \omega t)$ is periodic of period 2π with respect to $x_1, \ldots, x_l$ $(l \leqslant n)$, and $K = {}^1/_2(\mathbf{x}^\cdot, M(\omega t)\mathbf{x}^\cdot)$, $M(\omega t) = M(-\omega t)$, $V(\mathbf{x}, \omega t) = V(-\mathbf{x}, -\omega t)$. Let us investigate the solution $\mathbf{x}(t)$ of the following form:

$$x_i(t) = n_i\omega t + \psi_i(t),\ x_i(0) = 0,\ i = 1, \ldots, n \tag{3.1}$$

where $n_i = 0$ when $i > l$, n_i is any integer when $i \leqslant l$, and $\psi_i(t) = \psi_i(t + T)$ is the oscillating component of the solution. When $n_i = 0$, a periodic oscillation corresponds to the respective coordinate x_i, and when $n_i \neq 0$, a rotary motion with an average angular velocity $n_i\omega$ corresponds to it. Since the function $\mathbf{v}(t) = -\mathbf{x}(-t)$ also satisfies (2.1) and $\mathbf{v}(0) = \mathbf{x}(0)$, $\mathbf{v}^\cdot(0) = \mathbf{x}^\cdot(0)$, we have $\mathbf{x}(t) = -\mathbf{x}(-t)$.

Note that solutions of the form (3.1) are typical for systems containing angular coordinates.

Theorem 3. When $\omega > \omega_n^+$ and any given n_i (2.1) has a unique solution of the form (3.1).

Proof. We set $M(\omega t, \varepsilon) = M_- + \varepsilon[M(\omega t) - M_-]$, $V(\mathbf{x}, \omega t, \varepsilon) = \varepsilon V(\mathbf{x}, \omega t)$. Let $\mathbf{x}(t, \varepsilon)$ be the solution of the boundary value problem for (2.1) with conditions $x_i(0) = 0$, $x_i(T/2) = \pi n_i$, $i = 1, \ldots, n$. Obviously, the solution $\mathbf{x}(t, \varepsilon)$ continued in t is of the form (3.1), $\mathbf{x}(t, 1) = \mathbf{x}(t)$. The respective boundary value problem for the variational equation has the form

$$[M(\omega t, \varepsilon)\,\mathbf{z}^\cdot]^\cdot + \lambda U(\mathbf{x}(t, \varepsilon), \omega t, \varepsilon)\,\mathbf{z} = 0,\ \mathbf{z}(0) = \mathbf{z}(T/2) = 0 \tag{3.2}$$

Since $M(\omega t, \varepsilon) \geqslant M_-$, $U(\mathbf{x}(t, \varepsilon), \omega t, \varepsilon) \leqslant U_+$ when $\varepsilon \in [0, 1]$, the first positive eigenvalue of problem (3.2) is $\lambda_1 \geqslant \lambda_1^+ = 4\pi^2(T\omega_n^+)^{-2} > 1$ when $\omega > \omega_n^+$, where λ_1^+ is the eigenvalue when $M = M_-$, $U = U_+$. Hence when $\lambda = 1$ and any $\varepsilon \in [0, 1]$, problem (3.2) has no non-trivial solutions.

Consequently, if a solution of the form (3.1) exists for some $\varepsilon_* \in [0, 1]$, it can be continued in a unique way with respect to ε in some neighbourhood of ε_* /7/. Hence the conditions indicated above /1/ that ensure the uniqueness of the continuation of the periodic solution $\mathbf{x}(t, \varepsilon)$ with respect to ε on $[0, 1]$ and, by the same token, that of the existence and uniqueness of $\mathbf{x}(t, 1) = \mathbf{x}(t)$, are also sufficient for solutions of the form (3.1).

Representing (2.1) in Hamiltonian form and setting

$$\varphi(t, \varepsilon) = (\boldsymbol{\psi}(t, \varepsilon) + M(\omega t, \varepsilon)\boldsymbol{\psi}^{\cdot}(t, \varepsilon),\ \boldsymbol{\psi}(t, \varepsilon) + M(\omega t, \varepsilon)\cdot\boldsymbol{\psi}^{\cdot}(t, \varepsilon))$$

we obtain, like the proof of Theorem 1, that in the case considered here, when $\varepsilon \in [0, 1]$, the estimates (1.7), (1.10) hold, and thus any solution $x(t, \varepsilon)$ is continuable on $(0, \infty)$, and for solutions (3.1) the estimate $(\varphi(0, \varepsilon), \varphi(0, \varepsilon)) \leqslant N$ holds. When $\varepsilon = 0$, (2.1) has obviously a unique solution $x_i(t, 0) = n_i\omega t\ (i \leqslant l),\ x_i(t, 0) = 0\ (i > l)$ of the form (3.1). The conditions that ensure the existence and uniqueness of solution $\mathbf{x}(t, 1) = \mathbf{x}(t)$ are thus satisfied.

Remark. Since the function $V(\mathbf{x}, \omega t)$ is periodic of period 2π with respect to $x_1, \ldots, x_l$ and $V(\mathbf{x}, \omega t) = V(-\mathbf{x}, -\omega t)$, then $V(\mathbf{x} + \mathbf{x}_0, \omega t) = V(-(\mathbf{x} + \mathbf{x}_0), -\omega t)$ where $x_{i0} = 0$, where $i > l$, and $x_{i0} = 0$ or $x_{i0} = \pi$ when $i \leqslant l$. Hence, when $\omega > \omega_n^+$, a unique periodic solution of form $x_i(t) = x_{i0} + n_i\omega t + \psi_i(t)$ exists. Thus for given mean velocities of rotation $n_i\omega\ (i = 1, \ldots, l)$, 2^l solutions of (2.1) exist which correspond to various $\mathbf{x}_0$.

Note that this theorem does not exclude other types of solution of frequency $\omega > \omega_n^+$.

In view of the periodicity of $V(\mathbf{x}, \omega t)$ with respect to $x_1, \ldots, x_l$, the matrix $U(\mathbf{x}, \omega t)$ is not positive definite. Theorem 2 may be used to investigate solutions of this type.

We shall show that for fairly large ω, solution (3.1) exists, even if the elements $(\mathbf{x}, \omega t)$ of the matrix U are unbounded when $\mathbf{x} \in R^n$. To do this, we will first determine the upper bounds of the solutions, which are also of independent interest.

Let the matrix $M(\omega t)$ be reduced to diagonal form, i.e. $M(\omega t) = \operatorname{diag}[m_1(\omega t), \ldots, m_n(\omega t)]$, and $0 < m_i^- \leqslant m_i(\omega t) \leqslant m_i^+$. For given n_i we write system (2.1) in the form

$$[m_i(\psi_i^{\cdot} + n_i\omega)]^{\cdot} + f_i(\psi_1, \ldots, \psi_n, t) = 0,\quad i = 1, \ldots, n \tag{3.3}$$

Let us assume that for certain constants $c_{ik} \geqslant 0$, $p_i > 0$ and $\omega > \omega_n^+$ the following inequalities hold:

$$|f_i(\psi_1, \ldots, \psi_n, t)| < \sum_{k=1}^{n} c_{ik}|\psi_k| + p_i|\sin\omega t|,\qquad i = 1, \ldots, n \tag{3.4}$$

Let ω_*^2 be the largest real eigenvalue of the matrix $M_-^{-1}C$ $(M_- = \operatorname{diag}(m_1^-, \ldots, m_n^-)$, and $C = \|c_{ik}\|)$, $\psi_i^+(t) = A_i^+\sin\omega t$ be the solution of the system

$$m_i^-\psi_i^{\cdot\cdot} + \sum_{k=1}^{n} c_{ik}\psi_k + p_i\sin\omega t = 0,\qquad i = 1, \ldots, n \tag{3.5}$$

Theorem 4. The estimate

$$|\psi_i(t)| < A_i^+\sin\omega t \text{ on } (0, \pi/\omega),\ i = 1, \ldots, n \tag{3.6}$$

holds when $\omega > \omega_*$.

Proof. We set $c_{ik}(\varepsilon) = \varepsilon c_{ik}$ in (3.5). Since $\omega_*(\varepsilon) = \varepsilon\omega_*$, then with $\varepsilon \in [0, 1]$ the solution $\psi_i^+(t, \varepsilon) = A_i^+(\varepsilon)\sin\omega t$ exists. We shall show that $A_i^+(\varepsilon) > 0$. When $\varepsilon = 0$ this inequality is satisfied. Let $A_k^+(\varepsilon_*) = 0$, $A_i^+(\varepsilon_*) \geqslant 0$ for some $\varepsilon_* \in [0, 1]$, but then $\psi_k^{+\cdot\cdot}(t, \varepsilon_*) \equiv 0$ which is impossible, since by virtue of the k-th equation of (3.5), we have $m_k^-\psi_k^{+\cdot\cdot}(t, \varepsilon_*) < 0$ on $(0, \pi/\omega)$.

Let $\boldsymbol{\psi}(t, \varepsilon)\,(\boldsymbol{\psi}(0, \varepsilon) = \boldsymbol{\psi}(\pi/\omega, \varepsilon) = 0,\ \varepsilon \in [0, 1])$ be the solution of the system

$$[m_i(\omega t, \varepsilon)(\psi_i^{\cdot} + n_i\omega)]^{\cdot} + \varepsilon f_i(\psi_1, \ldots, \psi_n, t) = 0,\quad i = 1, \ldots, n \tag{3.7}$$

where $m_i(\omega t, \varepsilon) = m_i^- + \varepsilon(m_i(\omega t) - m_i^-)$. When $\varepsilon = 0$, the validity of the estimates is evident. If they are not satisfied when $\varepsilon = 1$, then k, $t_1 \in [0, \pi/\omega]$ and $\varepsilon_* \in (0, 1]$ can be found such that $|\psi_k(t_1, \varepsilon_*)| = \psi_k^+(t_1)$, $|\psi_k^{\cdot}(t_1, \varepsilon_*)| = |\psi_k^{+\cdot}(t_1)|$, $|\psi_i(t, \varepsilon_*)| \leqslant \psi_i^+(t)$ $(i = 1, \ldots, n;\ t \in [0, \pi/\omega])$. Since

$$m_k(\omega t, \varepsilon_*) > m_k^-,\qquad \varepsilon_*|f_k(\psi_1(t), \ldots, \psi_n(t), t)| < \sum_{r=1}^{n} c_{kr}\psi_r^+(t) + p_k\sin\omega t \text{ in } (0, \pi/\omega)$$

comparison of the k-th equation of (3.5) with (3.7) shows that $|\psi_k(t, \varepsilon_*)| < \psi_k^+(t)$ on $(0, \pi/\omega)$, $|\psi_k^{\cdot}(0, \varepsilon_*)| < \psi_k^{+\cdot}(0)$, $|\psi_k^{\cdot}(\pi/\omega), \varepsilon_*| < -\psi_k^{+\cdot}(\pi/\omega)$. This contradiction proves the theorem.

We separate the region $\Omega \subset R^n$ in the neighbourhood $\boldsymbol{\psi} = 0$. Let the inequalities (3.4) and $U(\mathbf{x}(t), \omega t) \leqslant U_+$ be satisfied when $\boldsymbol{\psi} \in \Omega$. Since $A_i^+(\omega) \to 0$ as $\omega \to \infty$, we have $\boldsymbol{\psi}^+(t) \in \Omega$ for fairly large ω; consequently also $\psi(t, \varepsilon) \in \Omega$ when $\varepsilon \in [0, 1]$. Hence under the conditions

$\omega > \omega_*(\Omega)$, $\omega > \omega_n^+(\Omega)$ the unique solution considered here exists in the region Ω.

4. Let us investigate the effect of small dissipative forces on the stability of periodic oscillations. Consider the equation

$$\frac{d}{dt}\left[M(\omega t)\frac{d\mathbf{x}}{dt}\right] + \mu F(\omega t)\frac{d\mathbf{x}}{dt} + \frac{\partial V(\mathbf{x}, \omega t)}{\partial \mathbf{x}} = 0 \qquad (4.1)$$

where $F(\omega t) = F(\omega t + 2\pi)$ is a $n \times n$ matrix that is symmetric and positive definite, and μ is a small positive parameter.

When assume that when $\mu = 0$, (4.1) has the peridic solution $\mathbf{x}(t)$ of period T, and the respective variational equation (2.2) is highly stable (this occurs when any of the stability conditions derived above is satisfied). Then (2.2) has no periodic solutions that are of period T, since an indefinite multiplier $\rho = 1$ would correspond to them. Hence by Poincaré's theorem /8/, (4.1) has, for small μ, the unique solution $\mathbf{x}(t, \mu)$ such that $\mathbf{x}(t, 0) = \mathbf{x}(t)$. The respective variational equation has the form

$$[M(\omega t)\mathbf{z}^\cdot]^\cdot + \mu F(\omega t)\mathbf{z}^\cdot + U(\mathbf{x}(t, \mu), \omega t)\mathbf{z} = 0 \qquad (4.2)$$

Let $\alpha_{i0} = i\omega_{i0}$ be an r_i-multiple characteristic exponent of (2.2). The characteristic exponents of (4.2), which become α_{i0} when $\mu = 0$, may be represented in the form

$$\alpha_i^k(\mu) = \alpha_{i0} + \alpha_{1i}^k(\mu) + \alpha_{2i}^k(\mu), \qquad k = 1, \ldots, r_i$$

where $\alpha_{1i}^k(\mu)$ are determined by the perturbation of the solution $\mathbf{x}(t)$ and, consequently of the coefficients of (2.2) and $\alpha_{2i}^k(\mu)$ by the presence of the term $\mu F\mathbf{z}^\cdot$. The first of these factors does not disturb the Hamiltonian character of the system, hence the quantities $\alpha_{1i}^k(\mu)$ are imaginary. Let us obtain expressions for $\alpha_{2i}^k(\mu)$.

The solutions

$$\mathbf{z}_{ik}(t) = \exp(\alpha_{i0}t)\,\mathbf{u}_{ik}(t), \quad k = 1, \ldots, r_i$$

correspond to the characteristic exponent α_{i0}.

We set $\mathbf{y}_{ik} = \mathbf{z}_{ik} + M\mathbf{z}_{ik}^\cdot = \exp(\alpha_{i0}t)\,\mathbf{v}_{ik}(t)$ and normalize the functions $\mathbf{v}_{ik} = \mathbf{u}_{ik} + M(\mathbf{u}_{ik}^\cdot + \alpha_{i0}\mathbf{u}_{ik})$ by the condition

$$((\mathbf{v}_{ip}, \gamma_i iJ\mathbf{v}_{iq})) = \frac{1}{T}\int_0^T (\mathbf{v}_{ip}, \gamma_i iJ\mathbf{v}_{iq})\,dt = \delta_{pq} \qquad (4.3)$$

where $\delta_{pq} = 0$ when $p \neq q$, $\delta_{pp} = 1$. We have $\gamma_i = 1$ for the multiplier of the first kind and $\gamma_i = -1$ for those of the second kind.

Since simple elementary divisors correspond to a definite multiplier, the expansion of $\alpha_{2i}^k(\mu)$ in powers of μ has the form

$$\alpha_{2i}^k(\mu) = \beta_{ik}\mu + O(\mu^{1+q_{ik}^{-1}})$$

where under condition (4.3) the quantities β_{ik} are eigenvalues of the matrix σ with elements

$$\sigma_{pq} = ((B\mathbf{v}_{ip}, \gamma_i iJ\mathbf{v}_{iq})), \quad B = \operatorname{diag}(0, -FM^{-1})$$

and the quantities q_{ik} are multiples of the respective values of β_{ik}.

Taking into account that $|\exp(\alpha_{i0}T)| = 1$, $F' = F$, we obtain

$$\sigma_{pq} = -((F\mathbf{z}_{ip}^\cdot, \gamma_i i\mathbf{z}_{iq})) = ((\gamma_i i\mathbf{z}_{ip}^\cdot, F\mathbf{z}_{iq})) = -((\gamma_i i\mathbf{z}_{ip}, F\mathbf{z}_{iq}^\cdot)) = \bar{\sigma}_{qp}$$

i.e. σ is a Hermitian matrix and its eigenvalues are therefore real. If β_{ik} are negative for all i, the solution $\mathbf{x}(t, \mu)$, when μ is small, is asymptotically stable; however, if some β_{ik} is positive, then $\mathbf{x}(t, \mu)$ is unstable.

In the case of a second-order scalar equation, the solution is stable to a first approximation, and becomes asymptotically stable, when subjected to small dissipative forces /9/. We shall explain under what conditions a similar statement holds for the vector equation (4.1).

Suppose that $F = M$ (which physically means that the coefficients of dissipation are proportional to the inertial coefficients). Then

$$\sigma_{pq} = -((M\mathbf{z}_{ip}^\cdot, \gamma_i i\mathbf{z}_{iq})) = -\tfrac{1}{2}((\mathbf{v}_{ip}, \gamma_i iJ\mathbf{v}_{iq})) = -\tfrac{1}{2}\delta_{pq}$$

Thus here all $\beta_{ik} = -\frac{1}{2}$ and, consequently, for small μ the solution $\mathbf{x}(t, \mu)$ is asymptotically stable.

Let us assume that all the characteristic exponents α_{i0} are simple, M and F are constant matrices, and matrix $U(t)$ is nearly constant (the latter occurs, for instance, in the case of forced oscillations of weakly non-linear systems). Since the functions $\mathbf{u}_i(t)$ are also close to constants, when $F > 0$, $\gamma_i\omega_{i0} = |\omega_{i0}|$, hence

$$\beta_i = \sigma_{ii} = -((F(\mathbf{u}_i^{\cdot} + i\omega_{i0}\mathbf{u}_i), \gamma_i i\mathbf{u}_i)) \approx -|\omega_{i0}|((F\mathbf{u}_i, \mathbf{u}_i)) < 0$$

Thus, also, in this case $\mathbf{x}(t, \mu)$ is asymptotically stable when μ is small.

5. As an example, let us consider the periodic oscillatory and rotary motions of a system of two connected pendulums (Fig.1). Let the potential energy of the system be $V_c = \frac{1}{2}c\rho^2$, where $\rho = |2R \sin \frac{1}{2}(x_1 + x_2)|$ is the distance between the mounting points of the two pendulums, and x_1, x_2 are the angular coordinates of the pendulums. The equations of forced oscillations of the system are

$$L_i x_i^{\cdot\cdot} + k_i \sin x_i + k \sin(x_1 + x_2) + F_i \sin \omega t = 0, \quad i = 1, 2 \tag{5.1}$$
$$L_i = m_i l_i^2, \ k_i = m_i g l_i, \ k = cR^2$$

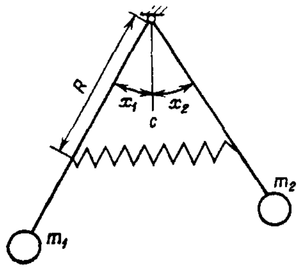

Fig.4

where m_i are the masses and l_i the lengths of pendulums, and g is the acceleration due to gravity.

The variational equation of the system motion is

$$M\mathbf{z}^{\cdot\cdot} + U(t)\mathbf{z} = 0 \tag{5.2}$$
$$M = \operatorname{diag}(L_1, L_2), \quad U(t) = \| u_{ik}(t) \|$$
$$u_{12} = u_{21} = k\cos(x_1(t) + x_2(t)), \ u_{ii} = k_i \cos x_i(t) + k\cos(x_1(t) + x_2(t))$$

Obviously $U(t) \leqslant U_+$, where U_+ is a matrix with elements $u_{ii}^+ = k_i + k$, $u_{12}^+ = u_{21}^+ = k$. Hence we can take as ω_i^{+2} the eigenvalues of the matrix $M^{-1}U_+$, i.e. the roots of the equation

$$\Delta(\omega) = (k_1 + k - L_1\omega^2)(k_2 + k - L_2\omega^2) - k^2 = 0$$

Note that the quantities ω^+_i are equal to the frequencies of small natural oscillations of the system.

In conformity with the remark to Theorem 3 (5.1), when $\omega > \omega_2^+$ there is a unique periodic solution of the form

$$x_i(t) = x_{i0} + n_i\omega t + \psi_i(t), \ \psi_i(t) = -\psi_i(-t) = \psi_i(t + T)$$

where (n_i are any given integers and $x_{i0} = 0$ or $x_{i0} = \pi$). When $n_1 = n_2 = 0$, the motion of the system is oscillatory and, when $x_{i0} = 0$, the oscillations of the respective pendulum occur realtive to the lower equilibrium position and when $x_{i0} = \pi$ they occur relative to the upper equilibrium position. The over-all number of oscillating motions of this type is four.

First, we shall investigate the periodic oscillations of the system in the neighbourhood of the equilibrium position ($n_1 = n_2 = x_{10} = x_{20} = 0$). The inequalities (3.4) will obviously be satisfied, if we set $C = U_+$, $p_i = |F_i|$. Hence when $\omega > \omega_2^+$, we have

$$|x_i(t)| < A_i^+ \sin \omega t \text{ on } (0, \pi/\omega) \tag{5.3}$$
$$A_1^+ = \frac{p_1(L_2\omega^2 - k_2 - k) + p_2 k}{\Delta(\omega)}, \quad A_2^+ = \frac{p_2(L_1\omega^2 - k_1 - k) + p_1 k}{\Delta(\omega)}$$

Since the function $-\mathbf{x}(t + T/2)$ also satisfies (5.1) and conditions (3.1), by virtue of the uniqueness, $\mathbf{x}(t) = -\mathbf{x}(t + T/2)$. In conformity with the remark to Theorem 2, the solution $\mathbf{x}(t)$ is stable when $\langle U\rangle > 0$ and $\omega > \omega_2^+$. In this system

$$(c, U(t)c) = k_1 \cos x_1(t) c_1^2 + k_2 \cos x_2(t) c_2^2 + k\cos(x_1(t) + x_2(t)) \times (c_1 + c_2)^2$$

Hence $U(t) > U^-$, $\langle U\rangle > \langle U^-\rangle$, when $A_1^+ + A_2^+ \leqslant \pi$, where $U^-(t)$ is a matrix obtained from $U(t)$ by replacing $x_i(t)$ by their upper bounds (5.3). The conditions of positive definiteness $\langle U^-\rangle$ have the form

$$k_1 J_0(A_1^+) + kJ_0(A_1^+ + A_2^+) > 0 \tag{5.4}$$
$$k_1 k_2 J_0(A_1^+) J_0(A_2^+) + kJ_0(A_1^+ + A_2^+)[k_1 J_0(A_1^+) + k_2 J_0(A_2^+)] > 0$$
$$J_0(A) = \frac{1}{\pi}\int_0^\pi \cos(A \sin t)\, dt$$

where $J_0(A)$ is a Bessel function of the first kind.

Thus when $A_1^+ + A_2^+ \leqslant \pi$, under conditions (5.4) the solution $\mathbf{x}(t)$ is stable to a first approximation. Using (5.3) it is possible to determine ω_* at which these inequalities begin to be satisfied.

The estimates (5.3) and consequently, the conditions of stability (5.4) obviously hold for any perturbing forces $F_i(t)$ of the form

$$F_i(t) = -F_i(-t) = -F_i(t + T/2), \quad |F_i(t)| \leqslant p_i |\sin \omega t|$$

Let us now investigate the solution of the form (3.1), when $n_1 = -1, n_2 = 1$ that defines the rotation of pendulums in one and the same direction at the average angular velocity ω. Let us assume that

$$R_1 = F_1 - m_1 g l_1 > 0, \ R_2 = F_2 + m_2 g l_2 < 0$$

and, first, consider the disconnected pendulums $(c = 0)$.

Since $\sin(\psi_1 - \omega t) = -\sin \omega t + \theta(t)\psi_1$, where $-1 \leqslant \theta(t) \leqslant 1$, setting $c_{11} = m_1 g l_1$ and using Theorem 4, we obtain $|\psi_1(t)| \leqslant R_1 [L_1\omega^2 - k_1]^{-1} \sin \omega t$ on $(0, \pi/\omega)$ when $\omega^2 > g/l_1$. We shall show that $\psi_1(t) > 0$ on $(0, \pi/\omega)$. We set $\theta(t, \varepsilon) = \varepsilon\theta(t)$; when $\varepsilon = 0$ the inequality indicated is satisfied and $\psi_1^{\cdot}(0) > 0$, $\psi_1^{\cdot}(\pi/\omega) < 0$. If it is violated when ε increases on $(0, 1]$, then $\psi_1(t_1, \varepsilon_*) = \psi_1^{\cdot}(t_1, \varepsilon_*) = 0$, $\psi_1(t, \varepsilon_*) \geqslant 0$ on $[0, \pi/\omega]$ for some $t_1 \in [0, \pi/\omega]$ and $\varepsilon_* \in (0, 1]$. But by virtue of the respective equation we have $\psi_1(t, \varepsilon_*) < 0$ in the neighbourhood of t_1. This contradiction proves that $\psi_1(t) > 0$ on $(0, \pi/\omega)$.

If $\psi_1(t) \leqslant \pi/2$ on $(0, \pi/\omega)$, then taking $\psi_1(t) > 0$ into account, we have

$$|x_1(t)| \leqslant |x_1(\pi/\omega - t)|, \quad \cos x_1(t) \geqslant |\cos x_1(\pi/\omega - t)|$$

Consequently the mean value of $\cos x_1(t)$ on $(0, \pi/\omega)$ is positive. Thus in conformity with Theorem 2 and motion considered here is stable, if

$$\frac{R_1}{L_1\omega^2 - k_1} \leqslant \frac{\pi}{2}, \quad \omega > 2\left(\frac{g}{l_1}\right)^{1/2}$$

The rotation may then be very irregular, it is even possible that $x_1^{\cdot}(0) > 0$, i.e. in some interval the pendulum rotation is reversed (note that the usual tests of rotation stability of the pendulum are inapplicable to such modes /11, 12/). The motion of the second pendulum is obviously stable, when

$$\frac{|R_2|}{L_2\omega^2 - k_2} \leqslant \frac{\pi}{2}, \quad \omega > 2\left(\frac{g}{l_2}\right)^{1/2}$$

Now let $c \neq 0$. To obtain the upper bounds of $|\psi_i(t)|$ it is obviously necessary to replace p_i by $|R_i|$ in (5.3). Let us assume that beginning with some $\omega_* > 2\omega_2^+$ the inequalities

$$A_1^+ \leqslant \pi/2, \quad A_2^+ \leqslant \pi/2, \quad R_1 > cR^2A_2^+, \quad R_2 < -cR^2A_1^+$$

are satisfied.

We shall show that the respective solution is stable. By virtue of the first of equations (5.1), the inequality $\psi_1(t) > 0$ is not violated on $(0, \pi/\omega)$ for some $c_* \leqslant c$, since

$$R_1 \sin \omega t + c_* R^2 \sin(\psi_1 + \psi_2) > (R_1 - c_* R^2 A_2^+) \sin \omega t > 0 \text{ on } (0, \pi/\omega)$$

From the second of equations (5.1) we similarly find that $\psi_2(t) < 0$ on $(0, \pi/\omega)$. As the result, the mean value of $\cos x_1(t)$, $\cos x_2(t)$, $\cos(x_1(t) + x_2(t))$ and, consequently, that of the matrix $U(t)$ on $[0, \pi/\omega]$ is positive, which, using Theorem 2, proves the stability of the solution considered.

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